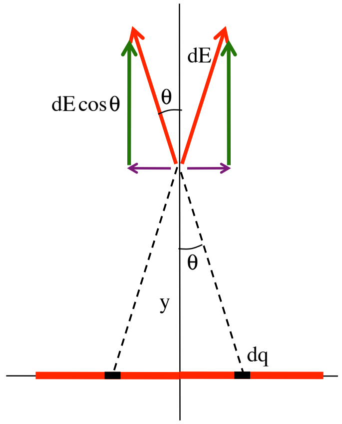
Problem 23.37

A uniformly charged rod will have an electric field function derives as follows:

To begin with, notice that the electric field component from different pieces of differential charge "dq" on opposite sides of the rod's axis will have horizontal components that add to zero (see sketch). That means we only have to determine the net field due to the vertical components.

(Note that I'm not using "d" for the distance to the point along the "y" axis because I'm also using "d" to denote differential quantities.)



The differential length "dx" on the rod will have a differential charge "dq" on it equal to:

$$dq = \lambda dx$$

With that and the fact that

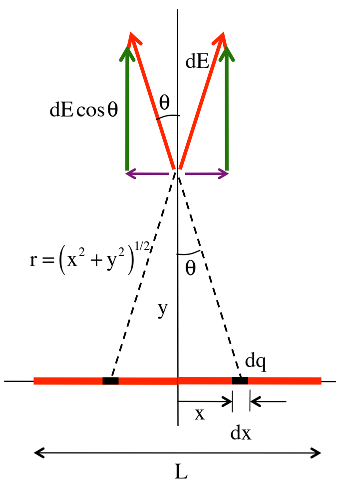
$$\cos\theta = \frac{y}{r} = \frac{y}{\left(x^2+y^2\right)^{1/2}}$$

we can write:

$$dE_y = dE\cos\theta = \quad k\frac{dq}{r^2} \quad \cos\theta$$

$$= \left(k\frac{(\lambda dx)}{\left(\left(x^2+y^2\right)^{1/2}\right)^2}\right)\left(\frac{y}{\left(x^2+y^2\right)^{1/2}}\right)$$

$$= (k\lambda)\left(\frac{y}{\left(x^2+y^2\right)^{3/2}}dx\right)$$



It would be nice to integrate over the rod, but because it's an odd function, we will instead integrate from zero to L/2, then double it. Doing so yields:

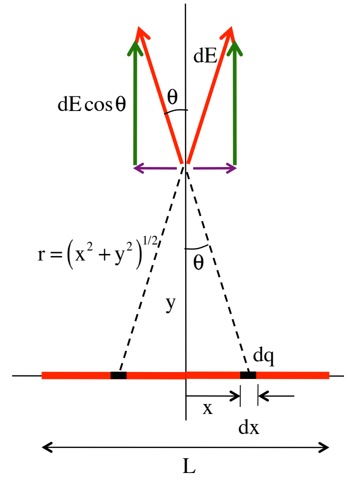
$$|\vec{E}| = \int dE_y$$

$$= 2(k\lambda)\int_{x=0}^{L/2}\frac{y}{\left(x^2+y^2\right)^{3/2}}dx$$

$$= 2(k\lambda y)\left[\frac{1}{y^2}\frac{x}{\left(x^2+y^2\right)^{1/2}}\Big|_{x=0}^{L/2}\right]$$

$$= 2(k\lambda)\frac{1}{y}\left[\frac{L/2}{\left(\left(L/2\right)^2+y^2\right)^{1/2}} - \frac{0}{\left(0+y^2\right)^{1/2}}\right]$$

$$= 2(k\lambda)\frac{1}{y}\left[\frac{L/2}{\left(\left(L/2\right)^2+y^2\right)^{1/2}}\right] \quad \text{(this will be in the } +\hat{j} \text{ direction)}$$

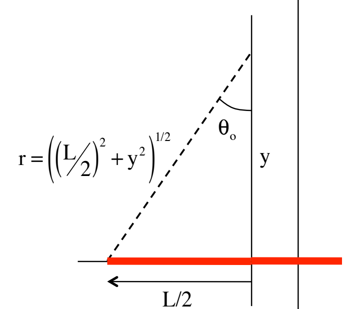


From the sketch, we can see that:

$$\sin\theta_o = \frac{L/2}{\left(\left(L/2\right)^2+y^2\right)^{1/2}}$$

So matching this with our derived expression (and ignoring the constants out in front), the question is, is:

$$\frac{\sin\theta_o}{y} \overset{?}{=} \frac{1}{y}\left[\frac{L/2}{\left(\left(L/2\right)^2+y^2\right)^{1/2}}\right]$$



The answer is "yes" which means $|\vec{E}| = 2(k\lambda)\frac{1}{y}\left[\frac{L/2}{\left(\left(L/2\right)^2+y^2\right)^{1/2}}\right] = 2(k\lambda)\frac{\sin\theta_o}{y}$.

As for what happens when the rod gets infinitely long, the angle θ_o goes to 90° and

$$|\vec{E}| = 2(k\lambda)\frac{\sin 90^\circ}{y} = 2\frac{(k\lambda)}{y} .$$